

Exam Analysis on Manifolds

WBMA013–05

Tuesday 30.01.2024, 8:30 – 10:30

You are allowed to use your cheatsheet (handwritten A4 page) during the exam. Make sure to clearly explain the steps in your proofs and computations. The exam consists of two pages with a total of 4 exercises. You get 10 points for free.

Exercise 1. (10 + 10 + 5 points)

Consider the set $M = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3 - x\} \subseteq \mathbb{R}^2$.

1. Show that M is a smooth manifold. What is its dimension?
2. Compute the critical points and critical values of the projection map $\pi : M \rightarrow \mathbb{R}$ defined by $\pi(x, y) = x$.
3. The manifold M is the solution of a polynomial equation. If P is a polynomial in x and y with real coefficients, is the set $M = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}$ always a smooth manifold? Prove this or give a counterexample.

Exercise 2. (10 + 5 + 10 points)

1. Define and (r, s) -tensor and explain what it is and its properties in your own words. What is the difference between a tensor and a tensor field? How do these concepts relate with the concept of differential forms and metrics?
2. Let $g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \in \mathcal{T}_2^0(\mathbb{R}^2)$, where we are using the global coordinates (x^1, x^2) .
 - (a) find the value of $g(X, Y)$ for $X = X_i \frac{\partial}{\partial x^i}$ and $Y = Y_i \frac{\partial}{\partial x^i}$.
 - (b) Compute $\phi^* g$ for $\phi(r, \theta) = (r \cos \theta, r \sin \theta)$.

The exam continues on the back side.

Exercise 3. (8 points)

Let M be a smooth manifold. Prove that for all $X, Y \in \mathcal{X}(M)$ and for all $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

Exercise 4. (8 + 10 + 14 points)

Let $M = \mathbb{R}^{2n+1}$ as a smooth $(2n + 1)$ -dimensional manifold. Take on M the global coordinates (x^i, y_i, z) where $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Let $\eta \in \Omega^1(M)$ defined in coordinates as $\eta := dz - y_i dx^i$. Given $H \in C^\infty(M)$ define $X_H \in \mathcal{X}(M)$ by

$$\begin{cases} \iota_{X_H} \eta = -H \\ \iota_{X_H} d\eta = dH - dH\left(\frac{\partial}{\partial z}\right)\eta \end{cases} .$$

You can assume that a solution X_H always exists.

1. Show that $\mathcal{L}_{X_H} \eta = -dH\left(\frac{\partial}{\partial z}\right)\eta$, where $\mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X(d\omega)$ is the Lie derivative of a differential form ω along the vector field X .
2. Show that $(d\eta)^n \wedge \eta$ is a volume form on M . Here $\omega^n := \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$.
3. Let $\gamma : I \subset \mathbb{R} \rightarrow M$, $\gamma(t) = (x^i(t), y_i(t), z(t))$, be a curve solving $\gamma'(t) = (X_H)_{\gamma(t)}$. Show that γ satisfies the system of differential equations

$$\begin{cases} \dot{x}^j(t) = \frac{\partial H}{\partial y_j} \Big|_{\gamma(t)} \\ \dot{y}_j(t) = -\frac{\partial H}{\partial x^j} \Big|_{\gamma(t)} - y_j(t) \frac{\partial H}{\partial z} \Big|_{\gamma(t)} \\ \dot{z}(t) = y_j(t) \frac{\partial H}{\partial y_j} \Big|_{\gamma(t)} - H \Big|_{\gamma(t)} \end{cases} .$$

Hint: express X_H in the given coordinate basis in terms of $H(x, y, z)$ and its derivatives.